Wild hypersurface bundles over toric varieties

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Abstract

In this paper, we investigate when there exists a wild hypersurface bundle over a smooth proper toric variety in positive characteristic. In particular, we determine the possibilities for toric varieties with Picard number at most three or toric Fano varieties of dimension at most four. Moreover, we can construct wild hypersurface bundles over them.

1 Introduction

A *wild hypersurface bundle* is a peculiar phenomenon in positive characteristic (see Definition 3.1). Only few examples of wild hypersurface bundles are known. Saito [13] completely determined when a smooth Fano 3-folds with Picard number 2 has a wild conic bundle structre. As a generalization for this result, Mori-Saito [10] showed the following:

Theorem 1.1 (Mori-Saito [10]) *Let $f : X \to S$ be a wild hypersurface bundle of degree p, $d = \dim S$ and $\dim X = 2d-1$. If S is isomorphic to a direct product of projective spaces, then one of the following holds*:

(i) $S \simeq \mathbb{P}^d$ *and X is a smooth divisor of bidegree $(1,p)$ in $\mathbb{P}^d \times \mathbb{P}^d$.*

(ii) $p = 2$, $S \simeq (\mathbb{P}^1)^d$ *and X is a smooth divisor in* $Y = \mathbb{P}_S(\mathcal{O}_S \oplus \bigoplus_{i=1}^d p_i^*\mathcal{O}_{\mathbb{P}^1}(1))$ *such that $X \sim 2\xi$, where $p_i : S \to \mathbb{P}^1$ is the i-th projection and ξ is the tautological line bundle of $Y \to S$.*

In this paper, we consider the case where S is a smooth proper toric d-fold. Using the technique in Mori-Saito [10], we completely determine the possibilities for S when the Picard number of S is 2 or 3 (see Section 4), or S is a toric Fano d-fold with $d \leq 4$ (see Section 5). Moreover, we can construct wild hypersurface bundles for these cases.

The content of this paper is as follows: Section 2 is a section for preparation. We review the concepts of primitive collections and relations, and explicitly describe the fans for projective space bundles over toric varieties. In Section 3, we review the definition of wild hypersurface bundles. The combinatorial version of the key result in Mori-Saito [10] is given. In Section 4, we consider the case where the Picard number of S is 2 or 3.

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There exist two new classes which have wild hypersurface bundle structures. In Section 5, we consider the case where S is a toric Fano variety. In particular, we determine the toric Fano d-folds which have wild hypersurface bundle structures for $d \leq 4$. These Fano varieties are interesting from the viewpoint of the birational geometry.

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2 Preliminaries

This section is devoted to explaining some basic facts of the toric geometry. See Batyrev [2], [3], Fulton [7], Oda [11] and Sato [14] more precisely.

Let $S = S_\Sigma$ be a smooth proper toric d-fold associated to a fan Σ over an algebraically closed field. Let $\mathrm{G}(\Sigma)$ be the set of primitive generators of 1-dimensional cones in Σ. A subset $P \subset \mathrm{G}(\Sigma)$ is called a *primitive collection* if P does not generate a cone in Σ, while any proper subset of P generates a cone in Σ. We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ. For a primitive collection $P = \{x_1, \ldots, x_m\}$, there exists the unique cone $\sigma(P)$ in Σ such that $x_1 + \cdots + x_m$ is contained in the relative interior of $\sigma(P)$, since S is proper. So, we obtain an equality

$$x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n,$$

where $y_1, \ldots, y_n$ are the generators of $\sigma(P)$, that is, $\sigma(P) \cap \mathrm{G}(\Sigma) = \{y_1, \ldots, y_n\}$, and $b_1, \ldots, b_n$ are positive integers. We call this equality the *primitive relation* of P. Thus, we obtain an element $r(P)$ in $A_1(S)$ for any primitive collection $P \in \mathrm{PC}(\Sigma)$, where $A_1(S)$ is the group of 1-cycles on S modulo rational equivalences. We define the *degree* of P as $\deg P := (-K_S \cdot r(P)) = m - (a_1 + \cdots + a_n)$.

Proposition 2.1 (Batyrev [2], Reid [12]) *Let $S = S_\Sigma$ be a smooth projective toric variety. Then*

$$\mathrm{NE}(S) = \sum_{P \in \mathrm{PC}(\Sigma)} \mathbb{R}_{\geq 0} r(P),$$

where $\mathrm{NE}(S)$ is the Mori cone of S.

A primitive collection P is said to be *extremal* if $r(P)$ is contained in an extremal ray of $\mathrm{NE}(S)$. For the torus invariant curve C contained in this extremal ray, we have

$$N_{C/S} \simeq \mathcal{O}_C(1)^{\oplus(m-2)} \oplus \mathcal{O}_C^{\oplus(d-m-n+1)} \oplus \mathcal{O}_C(-b_1) \oplus \cdots \oplus \mathcal{O}_C(-b_n),$$

where $N_{C/S}$ is the normal bundle.

Next, we explain how to construct the fan corresponding to a projective space bundle over a toric variety.

Let $S = S_\Sigma$ be a smooth proper toric d-fold, Σ a fan in $N = \mathbb{Z}^d$, $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_l\}$ and $D_1, \ldots, D_l$ the torus invariant prime divisors corresponding to $x_1, \ldots, x_l$, respectively. For torus invariant divisors

$$E_1 = \sum_{i=1}^{l} c_{1,i} D_i,\ \ldots,\ E_r = \sum_{i=1}^{l} c_{r,i} D_i,$$

put

$$E = \mathcal{O} \oplus \mathcal{O}_S(E_1) \oplus \cdots \oplus \mathcal{O}_S(E_r).$$

We construct the fan $\widetilde{\Sigma}$ in $\widetilde{N} := N \oplus \mathbb{Z}^r$ corresponding to the $\mathbb{P}^r$-bundle $\mathbb{P}_S(E)$ over S.

Let $\{e_1, \ldots, e_r\}$ be the standard basis for $\mathbb{Z}^r$. The elements of $\mathrm{G}(\widetilde{\Sigma})$ are

$$y_1 := e_1,\ \ldots,\ y_r := e_r,\ y_{r+1} := -(e_1 + \cdots + e_r),$$

$$\widetilde{x}_1 := x_1 + \sum_{i=1}^{r} c_{i,1} e_i,\ \ldots,\ \widetilde{x}_l := x_l + \sum_{i=1}^{r} c_{i,l} e_i.$$

For a maximal cone $\sigma = \mathbb{R}_{\geq 0} x_{i_1} + \cdots + \mathbb{R}_{\geq 0} x_{i_d}$ in Σ, put $\widetilde{\sigma} := \mathbb{R}_{\geq 0} \widetilde{x}_{i_1} + \cdots + \mathbb{R}_{\geq 0} \widetilde{x}_{i_d} \subset \widetilde{N} \otimes \mathbb{R}$. Put $\widetilde{\tau}_i := \mathbb{R}_{\geq 0} y_1 + \cdots + \mathbb{R}_{\geq 0} y_{i-1} + \mathbb{R}_{\geq 0} y_{i+1} + \cdots + \mathbb{R}_{\geq 0} y_{r+1} \subset \widetilde{N} \otimes \mathbb{R}$ for $1 \leq i \leq r+1$. The set of maximal cones in $\widetilde{\Sigma}$ is

$$\{\widetilde{\sigma} + \widetilde{\tau}_i \mid \sigma \text{ is a maximal cone in } \Sigma,\ 1 \leq i \leq r+1\}.$$

The tautological line bundle ξ for $\mathbb{P}_S(E) \to S$ is $\mathcal{O}_{\mathbb{P}_S(E)}(F_{r+1})$, where F_{r+1} is the torus invariant prime divisor corresponding to y_{r+1}.

3 Wild hypersurface bundles

In this section, we review the definition of a *wild hypersurface bundle structure* and some results in Mori-Saito [10]. From now on, we work over an algebraically closed field k of characteristic $p > 0$.

Definition 3.1 (Mori-Saito [10]) Let X and S be smooth algebraic varieties over k, and $f : X \to S$ a projective flat morphism with M a relatively very ample divisor such that X is embedded in $\pi : \mathbb{P}_S(E) \to S$, where $E = f_* M$. We call f a *wild hypersurface bundle of degree* p if for any $s \in S$, the geometric fiber $f^{-1}(s)$ is defined in $\mathbb{P}_S(E)$ by $x^p = 0$ for some non-zero $x \in E_s$.

Let ξ be the tautological line bundle of $\mathbb{P}_S(E)$. Then, there exists a Cartier divisor L on S such that $X \sim p\xi + \pi^* L$ in $\operatorname{Pic} \mathbb{P}_S(E)$. Let $d = \dim S$. If $\dim X = 2d - 1$, then there exists an exact sequence

$$0 \to \mathcal{O}_S \to E^p \otimes L \to T_S \to 0, \tag{1}$$

where T_S is the tangent bundle of S (see Theorem 1 in Mori-Saito [10]). Thanks to this exact sequence, we can study wild hypersurface bundle structures easily. So, in this paper, we add the assumption $\dim X = 2d - 1$ to the definition of a wild hypersurface bundle of degree p. We will use these notation throughout this paper.

The following is a slight generalization of Proposition 5 in Mori-Saito [10]. The proof is similar.

Proposition 3.2 *Let $f : X \to S$ be a wild hypersurface bundle of degree p and C a normal rational curve on S such that*

$$T_S \otimes \mathcal{O}_C \simeq \bigoplus_{i=-\infty}^{2} \mathcal{O}_C(i)^{\oplus a_i}.$$

Then, the following hold.

(i) *If the restriction of the exact sequence* (1) *on C is non-split, then for any $a_i > 0$, $i-1$ is divisible by p.*

(ii) *If the restriction of the exact sequence* (1) *on C is split, then $p = 2$ and for any $a_i > 0$, i is an even number.*

Remark 3.3 In Proposition 3.2,

$$E^p \otimes L \otimes \mathcal{O}_C \simeq \left(\bigoplus_{i=-\infty}^{-1} \mathcal{O}_C(i)^{\oplus a_i} \right) \oplus \mathcal{O}_C^{\oplus a_0} \oplus \mathcal{O}_C(1)^{\oplus (a_1+2)} \oplus \mathcal{O}_C(2)^{\oplus (a_2-1)}$$

for the case (i), while

$$E^p \otimes L \otimes \mathcal{O}_C \simeq \left(\bigoplus_{i=-\infty}^{-1} \mathcal{O}_C(i)^{\oplus a_i} \right) \oplus \mathcal{O}_C^{\oplus (a_0+1)} \oplus \mathcal{O}_C(1)^{\oplus a_1} \oplus \mathcal{O}_C(2)^{\oplus a_2}$$

for the case (ii).

We apply this result for the case where S is a toric variety.

Corollary 3.4 *Let $S = S_\Sigma$ be a smooth proper toric d-fold and $f : X \to S$ a wild hypersurface bundle of degree p. For an extremal primitive relation*

$$x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n,$$

where $\{x_1, \ldots, x_m, y_1, \ldots, y_n\} \subset \mathrm{G}(\Sigma)$ and $b_1, \ldots, b_n$ are positive integers, one of the following holds.

(i) *$m + n = d + 1$ and $b_i + 1$ is divisible by p for any i.*

(ii) *$p = 2$, $m = 2$ and b_i is an even number for any i.*

Proof. This can be proven by Proposition 3.2 immediately. For the case (i), $N_{C/S}$ does not contained $\mathcal{O}_C$, so $m + n = d + 1$. The left part is similar. q.e.d.

Remark 3.5 For the case (i) in Corollary 3.4, let $\varphi : S \to \overline{S}$ be the associated extremal contraction. If $S \not\simeq \mathbb{P}^d$, then φ is birational and the image of the exceptional set of φ is a point.

4 Toric varieties with Picard number 2 or 3

In this section, we treat the case where S is a smooth proper toric d-fold with Picard number 2 or 3. We construct some examples of wild hypersurface bundles using the notion of *homogeneous coordinate rings* of toric varieties (see Cox [5]).

(I) The Picard number of S is two.

Proposition 4.1 *Let S be a smooth proper toric d-fold with Picard number 2. If there exists a wild hypersurface bundle $f : X \to S$, then $p = 2$ and S is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or*

$$\mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(2a-1)),$$

where a is a positive integer.

Proof. There exists a wild hypersurface bundle of degree 2 over $\mathbb{P}^1 \times \mathbb{P}^1$ (see Mori-Saito [10]). So, suppose $S \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$.

S is a $\mathbb{P}^m$-bundle over $\mathbb{P}^n$ by the classification of proper toric varieties with Picard number 2 (see Kleinschmidt [8]). On the other hand, $m = 1$ by the case (ii) in Proposition 3.2. Thus, $p = 2$ and $S \simeq \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(\alpha))$ for a non-negative integer α. Since the normal bundle $N_{C_1/S}$ of the torus invariant curve C_1 contained in another extremal ray is

$$\mathcal{O}_{C_1}(-\alpha) \oplus \mathcal{O}_{C_1}(1)^{\oplus(d-2)},$$

α is an odd number by the case (i) in Proposition 3.2. q.e.d.

Next, we construct a wild hypersurface bundle of degree 2 for the above case. So, let $S := \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(2a-1))$ for a positive integer a and Σ the associated fan. Then, the primitive relations of Σ are

$$\text{(a) } x_1 + \cdots + x_d = (2a-1)x_{d+1} \text{ and (b) } x_{d+1} + x_{d+2} = 0,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d+2}\}$. Let $D_1, \ldots, D_{d+2}$ be the torus invariant prime divisors corresponding to $x_1, \ldots, x_{d+2}$, respectively. We may assume that $\{x_1, \ldots, x_{d-1}, x_{d+1}\}$ is the standard basis for N. By considering the divisors of the rational functions corresponding to $x_1, \ldots, x_{d-1}, x_{d+1}$, we have $D_1 = \cdots = D_d$ and $D_{d+2} = (2a-1)D_1 + D_{d+1}$ in $\operatorname{Pic} S$. Let C_1 and C_2 be the torus invariant curves corresponding to the extremal primitive relations (a) and (b), respectively. Then, $(D_1 \cdot C_1) = 1$, $(D_{d+1} \cdot C_1) = -(2a-1)$, $(D_1 \cdot C_2) = 0$ and $(D_{d+1} \cdot C_2) = 1$. Put

$$E = \mathcal{O}_S^{\oplus d} \oplus \mathcal{O}_S((a-1)D_1 + D_{d+1}) \text{ and } L = \mathcal{O}_S(D_1).$$

Then, we can easily check that E and L satisfy the conditions

$$E^2 \otimes L \otimes \mathcal{O}_{C_1} = \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_1}(1)^{\oplus d} \text{ and } E^2 \otimes L \otimes \mathcal{O}_{C_2} = \mathcal{O}_{C_2}^{\oplus d} \oplus \mathcal{O}_{C_2}(2).$$

In fact, we can construct a wild hypersurface bundle for these E and L as follows.

Let $\widetilde{\Sigma}$ be the fan corresponding to $Y=\mathbb{P}_S(E)$. We use the same notation as in Section 2. The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_{d+1}+\widetilde{x}_{d+2}=y_1$, $y_1+\cdots+y_{d+1}=0$ and

$$\widetilde{x}_1+\cdots+\widetilde{x}_d=\begin{cases}(2a-1)\widetilde{x}_d+y_2+\cdots+y_{d+1} & \text{if} \quad a=1\\ (2a-1)\widetilde{x}_d+(a-2)y_1 & \text{otherwise,}\end{cases}$$

where $\mathrm{G}(\widetilde{\Sigma})=\{\widetilde{x}_1,\ldots,\widetilde{x}_{d+2},y_1,\ldots,y_{d+1}\}$. Let $\widetilde{D}_1,\ldots,\widetilde{D}_{d+2},F_1,\ldots,F_{d+1}$ be the torus invariant prime divisors corresponding to $\widetilde{x}_1,\ldots,\widetilde{x}_{d+2},y_1,\ldots,y_{d+1}$, respectively. Then, we have $\widetilde{D}_1=\cdots=\widetilde{D}_d$, $\widetilde{D}_{d+2}=(2a-1)\widetilde{D}_d+\widetilde{D}_{d+1}$, $F_2=\cdots=F_{d+1}$ and $F_{d+1}=(a-1)\widetilde{D}_1+\widetilde{D}_{d+1}+F_1$ in $\operatorname{Pic} Y$. Since the tautological line bundle ξ for $\pi:Y\to S$ is $\mathcal{O}_Y(F_{r+1})$, we have $X\sim 2\xi+\pi^*L=2F_{d+1}+\widetilde{D}_1=\widetilde{D}_{d+1}+\widetilde{D}_{d+2}+2F_1$. Thus, for example, the smooth hypersurface X in Y defined by the equation

$$X_{d+1}X_{d+2}Y_1^2+X_1Y_2^2+\cdots+X_dY_{d+1}^2=0$$

is a wild hypersurface bundle of degree 2 over S, where $X_1,\ldots,X_{d+2},Y_1,\ldots,Y_{d+1}$ are the homogeneours coordinates of Y corresponding to $\widetilde{D}_1,\ldots,\widetilde{D}_{d+2},F_1,\ldots,F_{d+1}$, respectively. We can easily check the smoothness of X, so we leave the details for the exercise.

(II) The Picard number of S is three.

In this case, we suppose $d\geq 3$.

Batyrev [2] classified smooth projective toric d-folds with Picard number 3 using the notion of primitive relations.

Theorem 4.2 (Batyrev [2]) *Let $S=S_\Sigma$ be a smooth projective toric d-fold with Picard number three. Then, one of the following holds.*

(i) $\#PC(\Sigma)=3$*, and for any distinct elements* $P_1,P_2\in\mathrm{PC}(\Sigma)$*, we have* $P_1\cap P_2=\emptyset$*.*

(ii) $\#\,\mathrm{PC}(\Sigma)=5$*, and there exists* $(p_0,p_1,p_2,p_3,p_4)\in(\mathbb{Z}_{>0})^5$ *such that* $p_0+p_1+p_2+p_3+p_4=d+3$ *and the primitive relations of* Σ *are*

$$v_1+\cdots+v_{p_0}+y_1+\cdots+y_{p_1}=c_2z_2+\cdots+c_{p_2}z_{p_2}+(b_1+1)t_1+\cdots+(b_{p_3}+1)t_{p_3},$$

$$y_1+\cdots+y_{p_1}+z_1+\cdots+z_{p_2}=u_1+\cdots+u_{p_4},\ z_1+\cdots+z_{p_2}+t_1+\cdots+t_{p_3}=0,$$

$$t_1+\cdots+t_{p_3}+u_1+\cdots+u_{p_4}=y_1+\cdots+y_{p_1} \text{ and}$$

$$u_1+\cdots+u_{p_4}+v_1+\cdots+v_{p_0}=c_2z_2+\cdots+c_{p_2}z_{p_2}+b_1t_1+\cdots+b_{p_3}t_{p_3},$$

where

$$\mathrm{G}(\Sigma)=\{v_1,\ldots,v_{p_0},y_1,\ldots,y_{p_1},z_1,\ldots,z_{p_2},t_1,\ldots,t_{p_3},u_1,\cdots,u_{p_4}\}$$

and $c_2,\ldots,c_{p_2},b_1,\ldots,b_{p_3}$ *are positive integers.*

For positive integers a and b, let $\Sigma^d(a,b)$ be the fan whose primitive relations are

$$x_1+\cdots+x_{d-1}=(2a-1)x_d+(2b-1)x_{d+2},\ x_d+x_{d+1}=0 \text{ and } x_{d+2}+x_{d+3}=0,$$

where $\mathrm{G}(\Sigma^d(a,b))=\{x_1,\ldots,x_{d+3}\}$ and $W^d(a,b)$ the associated toric d-fold with Picard number 3. The following Proposition holds.

Proposition 4.3 *Let S be a smooth proper toric d-fold with Picard number 3. If there exists a wild hypersurface bundle $f : X \to S$, then $p = 2$ and S is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $W^d(a, b)$.*

Proof. There exists a wild hypersurface bundle of degree 2 over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Mori-Saito [10]). So, suppose $S \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Suppose $\#PC(\Sigma) = 5$, that is, the case (ii) in Theorem 4.2. We use the same notation as in Theorem 4.2. First, we remark that the first, second and fourth primitive relations are extremal. By Corollary 3.4, we have $p_0+p_1+p_2-1+p_3 = p_1+p_2+p_4 = p_3+p_4+p_1 = d+1$. This is impossible.

So, we have $\#PC(\Sigma) = 3$. There exists a primitive relation $x_{d+2} + x_{d+3} = 0$ by Corollary 3.4, since S is projective. In particular, $p = 2$. Suppose that the types of the other extremal rays corresponding to the primitive relations P_1 and P_2 are small. Then, $\sigma(P_1) \cap P_2 \neq \emptyset$, $\sigma(P_1) \cap \{x_{d+2}, x_{d+3}\} \neq \emptyset$, $\sigma(P_2) \cap P_1 \neq \emptyset$ and $\sigma(P_2) \cap \{x_{d+2}, x_{d+3}\} \neq \emptyset$. This is impossible, because S must be a $\mathbb{P}^1$-bundle over a toric $(d-1)$-fold with Picard number 2. Thus, we have the primitive relation $x_d + x_{d+1} = 0$. It is obvious that the last primitive relation is $x_1 + \cdots + x_{d-1} = \alpha x_d + \beta x_{d+2}$. Moreover, by Corollary 3.4, α and β are odd numbers. q.e.d.

Let $S = W^d(a, b)$ and $D_1, \ldots, D_{d+3}$ be the torus invariant prime divisors corresponding to $x_1, \ldots, x_{d+3}$, respectively. Put

$$E \simeq \mathcal{O}_S^{\oplus(d-1)} \oplus \mathcal{O}_S((a-1)D_1 + D_{d+1}) \oplus \mathcal{O}_S((b-1)D_1 + D_{d+1}) \text{ and } L \simeq \mathcal{O}_S(D_1).$$

We can construct a wild hypersurface bundle for these E and L similarly as in the case (I).

Let $\widetilde{\Sigma}$ be the fan corresponding to $Y = \mathbb{P}_S(E)$. The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_d + \widetilde{x}_{d+1} = y_1$, $\widetilde{x}_{d+2} + \widetilde{x}_{d+3} = y_2$, $y_1 + \cdots + y_{d+1} = 0$ and

$$\widetilde{x}_1 + \cdots + \widetilde{x}_{d-1} = (2a-1)\widetilde{x}_d + (2b-1)\widetilde{x}_{d+2} + (a-1)y_1 + (b-1)y_2 + y_3 + \cdots + y_{d+1}$$

if $a = 1$ or $b = 1$, otherwise

$$\widetilde{x}_1 + \cdots + \widetilde{x}_{d-1} = (2a-1)\widetilde{x}_d + (2b-1)\widetilde{x}_{d+2} + (a-2)y_1 + (b-2)y_2,$$

where $\mathrm{G}(\widetilde{\Sigma}) = \{\widetilde{x}_1, \ldots, \widetilde{x}_{d+3}, y_1, \ldots, y_{d+1}\}$. Let $\widetilde{D}_1, \ldots, \widetilde{D}_{d+3}, F_1, \ldots, F_{d+1}$ be the torus invariant prime divisors corresponding to $\widetilde{x}_1, \ldots, \widetilde{x}_{d+3}, y_1, \ldots, y_{d+1}$, respectively. Then, we have $\widetilde{D}_1 = \cdots = \widetilde{D}_{d-1}$, $\widetilde{D}_{d+1} = (2a-1)\widetilde{D}_1 + \widetilde{D}_d$, $\widetilde{D}_{d+3} = (2b-1)\widetilde{D}_1 + \widetilde{D}_{d+2}$, $F_3 = \cdots = F_{d+1}$ and $F_{d+1} = (a-1)\widetilde{D}_1 + \widetilde{D}_d + F_1 = (b-1)\widetilde{D}_1 + \widetilde{D}_{d+2} + F_2$ in $\operatorname{Pic} Y$. Since the tautological line bundle ξ for $\pi : Y \to S$ is $\mathcal{O}_Y(F_{r+1})$, we have $X \sim 2\xi + \pi^*L = 2F_{d+1} + \widetilde{D}_1 = \widetilde{D}_d + \widetilde{D}_{d+1} + 2F_1 = \widetilde{D}_{d+2} + \widetilde{D}_{d+3} + 2F_2$. Thus, for example, the smooth hypersurface X in Y defined by the equation

$$X_d X_{d+1} Y_1^2 + X_{d+2} X_{d+3} Y_2^2 + X_1 Y_3^2 + \cdots + X_{d-1} Y_{d+1}^2 = 0$$

is a wild hypersurface bundle of degree 2 over S, where $X_1, \ldots, X_{d+3}, Y_1, \ldots, Y_{d+1}$ are the homogeneours coordinates of Y corresponding to $\widetilde{D}_1, \ldots, \widetilde{D}_{d+3}, F_1, \ldots, F_{d+1}$, respectively. We can easily check the smoothness of X, so we leave the details for the exercise.

5 Toric Fano varieties

In this section, we consider the case where S is a toric Fano d-fold. A *Fano* variety is a Gorenstein projective variety S whose anti-canonical divisor $-K_S$ is ample. We can easily check whether a given smooth projective toric variety is Fano or not using the notion of primitive collections and relations.

Proposition 5.1 (Batyrev [3], Sato [14]) *Let $S = S_\Sigma$ be a smooth projective toric variety. S is a Fano variety if and only if* $\deg P > 0$ *for any primitive collection $P \in$* $\mathrm{PC}(\Sigma)$.

Smooth toric Fano d-folds are classified for $d \leq 4$ (see Batyrev [1], [3], Oda [11], Sato [14] and Watanabe-Watanabe [15]). So, we determine the possibilities for these classified toric Fano varieties and construct wild hypersurface bundles over them.

The following Proposition is easy.

Proposition 5.2 *Let $f : X \to S$ be a wild hypersurface bundle over a toric Fano d-fold $S = S_\Sigma$ and $d \geq 3$. If there exists an extremal divisorial contraction $\varphi : S \to \overline{S}$, then*

$$S \simeq \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(2a-1))$$

for a positive integer a.

Proof. By Corollary 3.4, the image of the exceptional divisor of φ is a point. So, there exist exactly two cases by Bonavero's classification (see Bonavero [4]): (a) The Picard number of S is two, or (b) The Picard number of S is three and $\#PC(\Sigma) = 5$. However, the case (b) does not occur by Proposition 4.3. Thus, we complete the proof by Proposition 4.1. q.e.d.

Corollary 5.3 *Let $f : X \to S$ be a wild hypersurface bundle over a toric Fano d-fold $S = S_\Sigma$ and $d \geq 3$. Then, one of the following holds*:

(i) $S \simeq \mathbb{P}^d$.

(ii) $S \simeq (\mathbb{P}^1)^d$.

(iii) $S \simeq \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(2a-1))$ *for a positive integer a.*

(iv) *Every extremal contraction of S is either a $\mathbb{P}^1$-bundle structure or a small contraction, and there exists at least one small contraction.*

Proof. See Mori-Saito [10] for the cases (i) and (ii), and see the case (I) in Section 4 for the case (iii). So, suppose S is not one of them. For the case (i) in Corollary 3.4, we have $n \geq 2$ by Proposition 5.2. For the case (ii) in Corollary 3.4, we have $n = 0$ and the associated extremal contraction is a $\mathbb{P}^1$-bundle structure, since S is a Fano variety. q.e.d.

(I) $\dim S = 2$.

There exist exactly five toric del Pezzo surfaces

$$\mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right),\ S_6 \text{ and } S_7,$$

where S_6 and S_7 are the del Pezzo surfaces of degree 6 and 7, respectively. For any toric del Pezzo surface S, there exists a wild hypersurface bundle over S. In fact, for S_6 and S_7, we can construct wild hypersurface bundles similarly as in Section 4. We omit the precise calculation for these constructions, and use the same notation as in Section 4.

Example 5.4 Let $S = S_\Sigma$ be the del Pezzo surface S_7 of degree 7. The primitive relations are $x_1 + x_2 = x_3$, $x_1 + x_5 = 0$, $x_2 + x_4 = x_5$, $x_3 + x_4 = 0$ and $x_3 + x_5 = x_2$. We have $p = 2$. Put

$$E = \mathcal{O}_S \oplus \mathcal{O}_S(D_3) \oplus \mathcal{O}_S(D_5) \text{ and } L = \mathcal{O}_S(D_2).$$

The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_1 + \widetilde{x}_2 = \widetilde{x}_3 + y_2 + y_3$, $\widetilde{x}_1 + \widetilde{x}_5 = y_2$, $\widetilde{x}_2 + \widetilde{x}_4 = \widetilde{x}_5 + y_1 + y_3$, $\widetilde{x}_3 + \widetilde{x}_4 = y_1$, $\widetilde{x}_3 + \widetilde{x}_5 = \widetilde{x}_2 + y_1 + y_2$ and $y_1 + y_2 + y_3 = 0$. The hypersurface X in $Y = \mathbb{P}_S(E)$ defined by the equation

$$X_3X_4Y_1^2 + X_1X_5Y_2^2 + X_2Y_3^2 = 0$$

is a wild hypersurface bundle of degree 2 over S.

Example 5.5 Let $S = S_\Sigma$ be the del Pezzo surface S_6 of degree 6. The primitive relations are $x_1 + x_5 = 0$, $x_3 + x_4 = 0$, $x_2 + x_6 = 0$, $x_3 + x_6 = x_1$, $x_3 + x_5 = x_2$, $x_1 + x_2 = x_3$, $x_5 + x_6 = x_4$, $x_2 + x_4 = x_5$ and $x_1 + x_4 = x_6$. We have $p = 2$. Put

$$E = \mathcal{O}_S \oplus \mathcal{O}_S(D_5 - D_6) \oplus \mathcal{O}_S(-D_2 + D_4) \text{ and } L = \mathcal{O}_S(D_2 + D_3).$$

The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_1 + \widetilde{x}_5 = y_1$, $\widetilde{x}_3 + \widetilde{x}_4 = y_2$, $\widetilde{x}_2 + \widetilde{x}_6 = y_3$, $\widetilde{x}_3 + \widetilde{x}_6 = \widetilde{x}_1 + y_2 + y_3$, $\widetilde{x}_3 + \widetilde{x}_5 = \widetilde{x}_2 + y_1 + y_2$, $\widetilde{x}_1 + \widetilde{x}_2 = \widetilde{x}_3 + y_1 + y_3$, $\widetilde{x}_5 + \widetilde{x}_6 = \widetilde{x}_4 + y_1 + y_3$, $\widetilde{x}_2 + \widetilde{x}_4 = \widetilde{x}_5 + y_2 + y_3$, $\widetilde{x}_1 + \widetilde{x}_4 = \widetilde{x}_6 + y_1 + y_2$ and $y_1 + y_2 + y_3 = 0$. The hypersurface X in $Y = \mathbb{P}_S(E)$ defined by the equation

$$X_1X_5Y_1^2 + X_3X_4Y_2^2 + X_2X_6Y_3^2 = 0$$

is a wild hypersurface bundle of degree 2 over S.

(II) $\dim S = 3$.

There does not exist a small contraction from any smooth toric Fano 3-fold. Therefore, if there exists a wild hypersurface bundle over S, then S is isomorphic to one of the following by Corollary 5.3:

$$\mathbb{P}^3,\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } \mathbb{P}_{\mathbb{P}^2}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\right).$$

(III) $\dim S = 4$.

There exists a wild hypersurface bundle over S, if S is isomorphic to one of the following:

$$\mathbb{P}^4,\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}_{\mathbb{P}^3}\left(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)\right) \text{ and } \mathbb{P}_{\mathbb{P}^3}\left(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(3)\right).$$

So, suppose S is not one of them, that is, the case (iv) in Corollary 5.3. By the classification of smooth toric Fano 4-folds, there exist exactly four possibilities:

(i) $S \simeq W^4(1,1)$,

(ii) S is the toric Fano 4-fold of type M_1 (see Batyrev [3] and Sato [14]),

(iii) S is the 4-dimensional pseudo del Pezzo variety $\widetilde{V}^4$ (see Ewald [6]) and

(iv) S is the 4-dimensional del Pezzo variety V^4 (see Klyachko-Voskresenskij [9]).

The first case is studied in Section 4, and we can construct wild hypersurface bundles for the other cases similarly as in Section 4. We omit the precise calculation for these constructions, and use the same notation as in Section 4.

Example 5.6 Let $S = S_\Sigma$ be the toric Fano 4-fold of type M_1. The primitive relations are $x_1 + x_8 = 0$, $x_4 + x_5 = 0$, $x_6 + x_7 = 0$, $x_1 + x_2 + x_3 = x_4 + x_6$, $x_4 + x_6 + x_8 = x_2 + x_3$, $x_2 + x_3 + x_5 = x_6 + x_8$ and $x_2 + x_3 + x_7 = x_4 + x_8$. We have $p = 2$. Put

$$E = \mathcal{O}_S \oplus \mathcal{O}_S \oplus \mathcal{O}_S(D_8) \oplus \mathcal{O}_S(D_4) \oplus \mathcal{O}_S(D_6) \text{ and } L = \mathcal{O}_S(D_3).$$

The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_1 + \widetilde{x}_8 = y_1$, $\widetilde{x}_4 + \widetilde{x}_5 = y_2$, $\widetilde{x}_6 + \widetilde{x}_7 = y_3$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_3 = \widetilde{x}_4 + \widetilde{x}_6 + y_1 + y_4 + y_5$, $\widetilde{x}_4 + \widetilde{x}_6 + \widetilde{x}_8 = \widetilde{x}_2 + \widetilde{x}_3 + y_1 + y_2 + y_3$, $\widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_5 = \widetilde{x}_6 + \widetilde{x}_8 + y_2 + y_4 + y_5$, $\widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_7 = \widetilde{x}_4 + \widetilde{x}_8 + y_3 + y_4 + y_5$ and $y_1 + y_2 + y_3 + y_4 + y_5 = 0$. The hypersurface X in $Y = \mathbb{P}_S(E)$ defined by the equation

$$X_1 X_8 Y_1^2 + X_4 X_5 Y_2^2 + X_6 X_7 Y_3^2 + X_2 Y_4^2 + X_3 Y_5^2 = 0$$

is a wild hypersurface bundle of degree 2 over S.

Example 5.7 Let $S = S_\Sigma$ be the 4-dimensional pseudo del Pezzo variety $\widetilde{V}^4$. The primitive relations are $x_4 + x_9 = 0$, $x_1 + x_5 = 0$, $x_2 + x_6 = 0$, $x_3 + x_7 = 0$, $x_1 + x_2 + x_9 = x_7 + x_8$, $x_1 + x_3 + x_9 = x_6 + x_8$, $x_2 + x_3 + x_9 = x_5 + x_8$, $x_1 + x_2 + x_3 = x_4 + x_8$, $x_4 + x_5 + x_8 = x_2 + x_3$, $x_4 + x_6 + x_8 = x_1 + x_3$, $x_4 + x_7 + x_8 = x_1 + x_2$, $x_5 + x_6 + x_8 = x_3 + x_9$, $x_5 + x_7 + x_8 = x_2 + x_9$ and $x_6 + x_7 + x_8 = x_1 + x_9$. We have $p = 2$. Put

$$E = \mathcal{O}_S \oplus \mathcal{O}_S(D_1) \oplus \mathcal{O}_S(D_2) \oplus \mathcal{O}_S(D_3) \oplus \mathcal{O}_S(D_9) \text{ and } L = \mathcal{O}_S(D_8).$$

The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_4 + \widetilde{x}_9 = y_4$, $\widetilde{x}_1 + \widetilde{x}_5 = y_1$, $\widetilde{x}_2 + \widetilde{x}_6 = y_2$, $\widetilde{x}_3 + \widetilde{x}_7 = y_3$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_9 = \widetilde{x}_7 + \widetilde{x}_8 + y_1 + y_2 + y_4$, $\widetilde{x}_1 + \widetilde{x}_3 + \widetilde{x}_9 = \widetilde{x}_6 + \widetilde{x}_8 + y_1 + y_3 + y_4$, $\widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_9 = \widetilde{x}_5 + \widetilde{x}_8 + y_2 + y_3 + y_4$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_3 = \widetilde{x}_4 + \widetilde{x}_8 + y_1 + y_2 + y_3$, $\widetilde{x}_4 + \widetilde{x}_5 + \widetilde{x}_8 = \widetilde{x}_2 + \widetilde{x}_3 + y_1 + y_4 + y_5$, $\widetilde{x}_4 + \widetilde{x}_6 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_3 + y_2 + y_4 + y_5$, $\widetilde{x}_4 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_2 + y_3 + y_4 + y_5$, $\widetilde{x}_5 + \widetilde{x}_6 + \widetilde{x}_8 = \widetilde{x}_3 + \widetilde{x}_9 + y_1 + y_2 + y_5$, $\widetilde{x}_5 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_2 + \widetilde{x}_9 + y_1 + y_3 + y_5$, $\widetilde{x}_6 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_9 + y_2 + y_3 + y_5$ and $y_1 + y_2 + y_3 + y_4 + y_5 = 0$. The hypersurface X in $Y = \mathbb{P}_S(E)$ defined by the equation

$$X_1 X_5 Y_1^2 + X_2 X_6 Y_2^2 + X_3 X_7 Y_3^2 + X_4 X_9 Y_4^2 + X_8 Y_5^2 = 0$$

is a wild hypersurface bundle of degree 2 over S.

Example 5.8 Let $S = S_\Sigma$ be the 4-dimensional del Pezzo variety V^4. The primitive relations are $x_4 + x_{10} = 0$, $x_1 + x_5 = 0$, $x_2 + x_6 = 0$, $x_3 + x_7 = 0$, $x_8 + x_9 = 0$, $x_1+x_2+x_{10} = x_7+x_8$, $x_1+x_3+x_{10} = x_6+x_8$, $x_2+x_3+x_{10} = x_5+x_8$, $x_1+x_2+x_3 = x_4+x_8$, $x_1+x_9+x_{10} = x_6+x_7$, $x_2+x_9+x_{10} = x_5+x_7$, $x_3+x_9+x_{10} = x_5+x_6$, $x_1+x_2+x_9 = x_4+x_7$, $x_1+x_3+x_9 = x_4+x_6$, $x_2+x_3+x_9 = x_4+x_5$, $x_4+x_5+x_6 = x_3+x_9$, $x_4+x_5+x_7 = x_2+x_9$, $x_4+x_6+x_7 = x_1+x_9$, $x_5+x_6+x_7 = x_9+x_{10}$, $x_4+x_5+x_8 = x_2+x_3$, $x_4+x_6+x_8 = x_1+x_3$, $x_4+x_7+x_8 = x_1+x_2$, $x_5+x_6+x_8 = x_3+x_{10}$, $x_5+x_7+x_8 = x_2+x_{10}$ and $x_6+x_7+x_8 = x_1+x_{10}$. We have $p = 2$. Put

$$E = \mathcal{O}_S \oplus \mathcal{O}_S(D_1 - D_9) \oplus \mathcal{O}_S(D_2 - D_9) \oplus \mathcal{O}_S(D_3 - D_9) \oplus \mathcal{O}_S(D_{10} - D_9) \text{ and}$$

$$L = \mathcal{O}_S(D_8 + D_9).$$

The primitive relations of $\widetilde{\Sigma}$ are $\widetilde{x}_4 + \widetilde{x}_{10} = y_4$, $\widetilde{x}_1 + \widetilde{x}_5 = y_1$, $\widetilde{x}_2 + \widetilde{x}_6 = y_2$, $\widetilde{x}_3 + \widetilde{x}_7 = y_3$, $\widetilde{x}_8 + \widetilde{x}_9 = y_5$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_{10} = \widetilde{x}_7 + \widetilde{x}_8 + y_1 + y_2 + y_4$, $\widetilde{x}_1 + \widetilde{x}_3 + \widetilde{x}_{10} = \widetilde{x}_6 + \widetilde{x}_8 + y_1 + y_3 + y_4$, $\widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_{10} = \widetilde{x}_5 + \widetilde{x}_8 + y_2 + y_3 + y_4$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_3 = \widetilde{x}_4 + \widetilde{x}_8 + y_1 + y_2 + y_3$, $\widetilde{x}_1 + \widetilde{x}_9 + \widetilde{x}_{10} = \widetilde{x}_6 + \widetilde{x}_7 + y_1 + y_4 + y_5$, $\widetilde{x}_2 + \widetilde{x}_9 + \widetilde{x}_{10} = \widetilde{x}_5 + \widetilde{x}_7 + y_2 + y_4 + y_5$, $\widetilde{x}_3 + \widetilde{x}_9 + \widetilde{x}_{10} = \widetilde{x}_5 + \widetilde{x}_6 + y_3 + y_4 + y_5$, $\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_9 = \widetilde{x}_4 + \widetilde{x}_7 + y_1 + y_2 + y_5$, $\widetilde{x}_1 + \widetilde{x}_3 + \widetilde{x}_9 = \widetilde{x}_4 + \widetilde{x}_6 + y_1 + y_3 + y_5$, $\widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_9 = \widetilde{x}_4 + \widetilde{x}_5 + y_2 + y_3 + y_5$, $\widetilde{x}_4 + \widetilde{x}_5 + \widetilde{x}_6 = \widetilde{x}_3 + \widetilde{x}_9 + y_1 + y_2 + y_4$, $\widetilde{x}_4 + \widetilde{x}_5 + \widetilde{x}_7 = \widetilde{x}_2 + \widetilde{x}_9 + y_1 + y_3 + y_4$, $\widetilde{x}_4 + \widetilde{x}_6 + \widetilde{x}_7 = \widetilde{x}_1 + \widetilde{x}_9 + y_2 + y_3 + y_4$, $\widetilde{x}_5 + \widetilde{x}_6 + \widetilde{x}_7 = \widetilde{x}_9 + \widetilde{x}_{10} + y_1 + y_2 + y_3$, $\widetilde{x}_4 + \widetilde{x}_5 + \widetilde{x}_8 = \widetilde{x}_2 + \widetilde{x}_3 + y_1 + y_4 + y_5$, $\widetilde{x}_4 + \widetilde{x}_6 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_3 + y_2 + y_4 + y_5$, $\widetilde{x}_4 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_2 + y_3 + y_4 + y_5$, $\widetilde{x}_5 + \widetilde{x}_6 + \widetilde{x}_8 = \widetilde{x}_3 + \widetilde{x}_{10} + y_1 + y_2 + y_5$, $\widetilde{x}_5 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_2 + \widetilde{x}_{10} + y_1 + y_3 + y_5$, $\widetilde{x}_6 + \widetilde{x}_7 + \widetilde{x}_8 = \widetilde{x}_1 + \widetilde{x}_{10} + y_2 + y_3 + y_5$ and $y_1 + y_2 + y_3 + y_4 + y_5 = 0$. The hypersurface X in $Y = \mathbb{P}_S(E)$ defined by the equation

$$X_1X_5Y_1^2 + X_2X_6Y_2^2 + X_3X_7Y_3^2 + X_4X_{10}Y_4^2 + X_8X_9Y_5^2 = 0$$

is a wild hypersurface bundle of degree 2 over S.

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